

High-energy QCD as a topological field theory

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Abstract. We propose an identification of the conformal field theory underlying Lipatov's spin-chain model of high-energy scattering in perturbative QCD. It is a twisted $N = 2$ supersymmetric topological field theory, which arises as the limiting case of the $SL(2, R)/U(1)$ non-linear σ model that also plays a role in describing the Quantum Hall effect and black holes in string theory. The doubly-infinite set of non-trivial integrals of motion of the high-energy spin-chain model displayed by Faddeev and Korchemsky are identified as the Cartan subalgebra of a $W_\infty \otimes W_\infty$ bosonic sub-symmetry possessed by this topological theory. The renormalization group and an analysis of instanton perturbations yield some understanding why this particular topological spin-chain model emerges in the high-energy limit, and provide a new estimate of the asymptotic behaviour of multi-Reggeized-gluon exchange.

1 Introduction and summary

In the last few years, Lipatov [1] and others have developed a theory of high-energy scattering in perturbative QCD, based on the t-channel exchanges of Reggeized gluons interacting via s-channel gluons. In the large- N_c limit, the elastic scattering amplitude is related to eigenstates of Hamiltonians with nearest-neighbour interactions, that are holomorphic and antiholomorphic functions of the transverse coordinates, and whose eigenvalues determine the asymptotic behaviour in the high-energy limit. These Hamiltonians for the exchange of N_g Reggeized gluons can be written as:

$$H_{N_g} = \sum_{k=1}^{N_g} H_{k,k+1} \quad ; \quad \bar{H}_{N_g} = \sum_{k=1}^{N_g} \bar{H}_{k,k+1}. \quad (1)$$

One imposes periodic boundary conditions $H_{n,n+1} = H_{n,1}$ in the holomorphic sector, and analogously $\bar{H}_{n,n+1} = \bar{H}_{n,1}$ in the anti-holomorphic sector, where we use bars to denote the replacements $z \rightarrow \bar{z}$, etc. The two-particle interactions can be expressed in several equivalent forms:

$$\begin{aligned} H_{jk} &= P_j^{-1} \log(z_j - z_k) P_j + P_k^{-1} \log(z_j - z_k) P_k + 2\gamma_E, \\ &= 2 \log(z_j - z_k) + (z_j - z_k) \log(P_j P_k) (z_j - z_k)^{-1} \\ &\quad + 2\gamma_E, \\ &= \sum_{l=0}^{\infty} \left(\frac{2l+1}{l(l+1) - \hat{L}_{ik}^2} - \frac{2}{l+1} \right), \end{aligned} \quad (2)$$

where

$$P_i \equiv i \frac{\partial}{\partial z_i}, \quad \hat{L}_{ik}^2 \equiv (z_i - z_k)^2 P_i P_k, \quad (3)$$

and γ_E is the Euler constant. Lipatov conjectured [1] that the model was integrable, was able to solve the case of two-gluon exchange exactly, suggested that the general case could be solved using the Bethe Ansatz, and exhibited some non-trivial integrals of motion.

Faddeev and Korchemsky [2] have observed that the Lipatov Hamiltonians are just those of Heisenberg ferromagnets with non-compact spins $s = 0, -1$. This observation is prompted by the fact that Lipatov's kernel \hat{L}_{ij} can be represented as a Heisenberg interaction term

$$\hat{L}_{ij} = S_i \cdot S_j \quad (4)$$

among spin operators S_i at neighboring sites i, j of a chain, whose components are defined in the (anti-)holomorphic sector of impact parameter space as follows:

$$S_i^+ = z_i^2 \partial_i - s z_i \quad ; \quad S_i^- = -\partial_i \quad ; \quad S_i^3 = z_i \partial_i - s. \quad (5)$$

This identification enabled them to find a doubly-infinite set of non-trivial integrals of motion, verify the integrability of the model, and solve it in the $N_g = 2$ case using a generalized Bethe Ansatz [2]. However, they did not give any symmetry origin for the integrals of motion, and neither they nor Lipatov [1] has identified the specific two-dimensional field theory corresponding to this lattice model in the large- N_g limit.

Although the prototypical case $N_g = 2$ can teach us many important lessons, it cannot by itself control the high-energy behaviour of QCD, in particular because it does not respect unitarity. It is therefore of interest to extend the above-mentioned analyses of the $N_g = 2$ case to larger N_g . There has indeed been considerable work on the kernel for $N_g = 3$, in connection with the odderon in

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QCD [3], and the general case of large N_g has also been examined [4]. Transitions in the t channel between different numbers of gluons have also been analyzed [5]. It would be valuable to develop a more powerful approach to the analysis of the large- N_g limit, which should resemble a two-dimensional field theory. As a step towards this goal, in this paper we identify the two-dimensional conformal field theory underlying Lipatov's two-body spin-chain Hamiltonian in the continuum limit.

In Sect. 2, we use symmetry principles and the field-theoretical description [6] of analogous compact Heisenberg ferromagnetic spin chains as guides to this identification. It is well known that such systems correspond to compact $O(3)$ non-linear σ models in the unitary condensed-matter cases $s = 1/2, 1, 3/2, \dots$. When the number of spin carriers at each site is fixed, such spin chains possess a local $U(1)$ symmetry, and we demonstrate that this is also a property of Lipatov's scattering kernel [1] for Reggeized gluons and of his effective Hamiltonian.

We argue in Sect. 3 that the conformal field theory corresponding to high-energy QCD is the limiting case $s \rightarrow 0^-$ (and $s \rightarrow -1^+$) of the non-compact $SO(2,1)/U(1)$ non-linear σ model, which describes a Heisenberg system with $s < 0$, and is known to possess a W_∞ symmetry. The doubly-infinite set of conserved quantities exhibited by Faddeev and Korchemsky [2] is the Cartan subalgebra of a $W_\infty \otimes W_\infty$ symmetry that appears in the continuum limit. This is a bosonic subalgebra of a twisted $N = 2$ supersymmetric W algebra possessed by the topological theories that are the limits of the non-compact non-linear σ models when $s \rightarrow 0, -1$, which correspond in the continuum limit to high-energy scattering in perturbative QCD. Quantum Hall conductors [7] and stringy black holes [8] are known to be described by the same non-compact non-linear σ model for generic values of the Landau level filling parameter ν (black hole mass), and to possess this enhanced $N = 2$ supersymmetry in the limit of complete filling $\nu = -1/s = 1$ (at the core of the black hole).

As we show in Sect. 4, instantons play an important rôle in the renormalization-group flow that drives the non-compact σ model towards the limiting case $s \rightarrow 1$. They also provide us with a parametric estimate of the dependence of the high-energy behaviour of the exchange of a large number of Reggeized gluons, corresponding to a cylindrical topology for the system exchanged in the t channel. More complicated topologies for the exchanged system could also be treated within this field-theoretical approach, but are not discussed in this paper.

Finally, in Sect. 5 we summarize our conclusions, emphasize the aspects in which our analysis stands in need of confirmation, and appraise some of the prospects for future progress.

2 Symmetries of Lipatov's spin-chain model

It will be convenient for our subsequent discussion to expand the Hamiltonian (2) formally as an infinite series in

powers of the Heisenberg operator (4) $S_i \cdot S_j$:

$$H_{ij} = -\frac{1}{S_i \cdot S_j} + \text{const} + \sum_{l=0}^{\infty} \frac{2l+1}{l^2(l+1)^2} (S_i \cdot S_j) + O[(S_i \cdot S_j)^2], \quad (6)$$

where the omitted operators are higher powers of the Heisenberg operator. They constitute irrelevant operators in a renormalization group sense, by naive power counting, so do not affect the continuum limit of the theory, and we shall not deal explicitly with them in what follows. However, (6) does contain a non-analytic term $1/S_i \cdot S_j$, due to the $l = 0$ partial wave in (2), for which the Taylor expansion fails. This is a particular feature of the non-compact spin formalism, and should be contrasted to the conventional case of compact spin $s > 0$, where Heisenberg chains can be represented as non-singular functions of $S_i \cdot S_j$. Formally, as we shall describe below, one can regularize the $l = 0$ limit by representing the finite-size spin chain, which corresponds to a fixed number of gluons N_g , as an infinite-size lattice chain with 'holes', i.e. missing spins. As a quantum-mechanical problem, removing spins is a complicated procedure, since it involves modification of the Hilbert space. However, attempts have been made to describe doping in anti-ferromagnetic chains with the hope of understanding the relevance of the hole dynamics in scenarios for magnetic superconductivity [9,10]. Below we borrow from these techniques.

An important feature of the model (1, 2) is the fact that the number of Reggeized gluons per lattice site is fixed. This implies that there is a local gauge symmetry in the Heisenberg interaction $S_i \cdot S_j$, which simply expresses the particle-number conservation law. This symmetry can be seen straightforwardly if we represent the Heisenberg interaction in terms of fundamental 'Reggeon' creation and annihilation operators $C_{\alpha,i}^\dagger$ and $C_{\alpha,i}$, in direct analogy with the corresponding representation in the solid-state models relevant to the description of high-temperature superconductivity [10]. To this end, we write the Heisenberg spin-spin interaction in a 'microscopic' form

$$S_i \cdot S_j = -J \sum_{\langle ij \rangle} \sum_{\alpha, \beta} T_{i, \alpha \beta} T_j^{\alpha \beta} \quad ; \quad T_i^{\alpha \beta} \equiv C_i^{\dagger, \alpha} C_i^\beta \quad (\text{no sum over } i), \quad (7)$$

where α, β denote spin s indices, i, j are lattice-site indices and $\langle \dots \rangle$ denote nearest-neighbour sites. To exhibit the $U(1)$ symmetry we introduce a *slave-boson* Ansatz

$$C_i^\alpha = \psi_{i, \alpha} z_i^\dagger, \quad (8)$$

where $\psi_{\alpha, i}, \psi_{\alpha, i}^\dagger$ are fermion operators that annihilate or create a 'hole' at a site i in the spin representation s . They satisfy canonical commutation relations. On the other hand, the z, z^\dagger are Bose fields that are spin singlets. The Ansatz (8) satisfies trivially a local $U(1)$ phase symmetry

$$\psi_{j, \alpha} \rightarrow e^{i\theta_j} \psi_{j, \alpha}, \quad z_j \rightarrow e^{i\theta_j} z_j, \quad (9)$$

which is a consequence of the local constraint restricting the number of Reggeons per site:

$$C_i^\dagger C_i = \psi_i^\dagger \psi_i + z^\dagger z = 2s. \quad (10)$$

It can be shown [9, 10] that the above formalism is a convenient way of representing the effects of holes in a spin chain as a path integral over fermionic variables parametrizing the Berry-phase term that describes the missing-spin effect in the action. Essentially, one describes the effects of the missing spin by subtracting from the action the contribution that a spin would make if it were there. This is a simple way of describing correctly the change in the Hilbert space.

By implementing the slave-fermion Ansatz, one can consider a situation where the total number of Reggeons is fixed at N_g , but the size of the lattice chain is infinite. This fixed-Reggeon-number case has a natural doping interpretation and the infinite-chain limit provides a field-theory interpretation in the continuum limit, as we discuss in more detail in Sect. 3. The ‘doping concentration’ can be defined formally as the vacuum expectation value η of the fermion bilinears in a splitting of the form:

$$\begin{aligned} \psi_{\alpha,i}^\dagger \psi_{\alpha,i} &= \langle \psi_{\alpha,i}^\dagger \psi_{\alpha,i} \rangle + : \psi_{\alpha,i}^\dagger \psi_{\alpha,i} : \\ &\equiv \eta + : \psi_{\alpha,i}^\dagger \psi_{\alpha,i} : . \end{aligned} \quad (11)$$

The splitting (11) provides us with the adverbized regularization of the non-analytic terms $1/S_i \cdot S_j$ in (6). Using the constraint (10) and the appropriate free-fermion commutation relation, the Heisenberg terms can be written in the form:

$$\begin{aligned} S_i \cdot S_j &= \eta + : \psi_{\alpha,i}^\dagger \psi_{\alpha,i} : \\ &\quad - 4s^2 \psi_{\alpha,j} \psi_{\alpha,i}^\dagger \psi_{\beta,i} \psi_{\beta,j}^\dagger + \dots, \end{aligned} \quad (12)$$

where the \dots indicate terms that are irrelevant operators in a renormalization group sense, by naive power counting. One can expand (12) formally in a Taylor series in powers of $4s^2/\eta$, and at the end one can take the twin limit $\eta, s \rightarrow 0$ to recover the ‘half-filled’ action (2). Clearly, the crucial test of the validity of this ‘hole-regulator’ scheme will be provided by a careful study of the scaling properties of the model, as a function of the doping concentration η . This is left for future work. What we argue below is that the above scheme provides one with the necessary tools to study the symmetries of the model in a straightforward and physical way.

Concentrating on the relevant operators (by naive power counting) one observes that the gauge-invariant effective Hamiltonian thus constructed contains amongst others a term

$$\sum_{ij} J_{ij} \psi_{i,\alpha}^\dagger \psi_j^\alpha \Delta_{ij} + \dots, \quad (13)$$

where $J_{ij} = (4s^2/\eta) + \sum_{l=0}^{\infty} 2l + 1/[l^2(l+1)^2]$ and the link variable $\Delta_{ij} = \langle \psi_i^{\alpha,\dagger} \psi_{\alpha,j} \rangle$ is a Hubbard-Stratonovich gauge field. For details of this construction we refer the

interested reader to the relevant literature [10]. Thus, the Hamiltonian can be written in terms of the field

$$M_{ij}^{\alpha\beta} \equiv \psi_i^{\alpha\dagger} \psi_j^\beta, \quad (14)$$

which is manifestly gauge invariant in the light-cone gauge, for which $\Delta_{ij} = 1$ along the links. This is a particularity of a two-dimensional gauge theory, which will provide us with the generators of an infinite-dimensional Lie algebra of symmetries, as we shall see later.

The basic advantage of the $U(1)$ gauge symmetry is that it allows for a parafermion construction of the conformal field theory corresponding to the continuum limit of the statistical model. This is important because the non-compact spin case of Lipatov’s kernel apparently corresponds to the limiting case of a non-compact $SL(2, R)$ Wess-Zumino σ model, as we discuss in Sect. 3. The extra $U(1)$ symmetry at any doping concentration, i.e. arbitrary but fixed Reggeon number with at most one Reggeon present per site, implies that one can mod out the local Abelian phase factors to obtain an $SL(2, R)/U(1)$ coset model. Such models are equivalent to parafermion models, and are known [11] to possess an infinite-dimensional W_∞ algebra of symmetries. Below we construct such symmetries explicitly in the lattice model.

Before doing so, we remark that Heisenberg chains with holes are known [12] to possess graded (supersymmetric) algebras generated by particles (spins) and holes (superpartners). To introduce the empty sites, one allows for a hopping element

$$t_{ij} \psi_i^\dagger \psi_j \quad (15)$$

in the Hamiltonian of the chain¹. At half-filling, $t_{ij} \rightarrow 0$, and this limit can be taken at the very end of our computation if necessary, after obtaining important symmetry information, e.g. on supersymmetry. In terms of projection operators $\chi^{AB} \equiv |A\rangle\langle B|$ on the $2s+1$ states on a lattice site, including the empty ones, the Hamiltonian reads:

$$H = \sum_{\sigma} \sum_{ij} (t_{ij} \chi_i^{0\sigma} \chi_j^{\sigma 0} + J_{ij} \chi_i^{\sigma\sigma'} \chi_j^{\sigma\sigma'}), \quad (16)$$

where the indices σ denote spin states, with the exclusion of the empty ones.

The operators χ^{AB} satisfy a supersymmetry algebra in the spin s representation. In the non-compact $s = -1$ case of interest such a supersymmetry becomes twisted, and the holes represent ghost states. Supersymmetry implies a $W_\infty \otimes W_\infty$ algebra in the bosonic sector [13]. A similar $W_\infty \otimes W_\infty$ structure arises in the quantum Hall system: one W_∞ appears at each Landau level, and is associated with magnetic translation operators, while the other mixes the various levels, and is associated with operators that appear in the Hamiltonian of the model [7]. In the Quantum Hall case there is also an associated supersymmetry, similar to that of the Heisenberg chain, which

¹ In the gauge-invariant formalism discussed above, this hopping element is induced by the doping, and is accompanied by the replacement $t_{ij} \rightarrow \Delta_{ij} t_{ij}$

explains the underlying $W_\infty \otimes W_\infty$ structure. The similarity of the Hall system to the present model is discussed further in the next section, where it is argued that both models can be mapped onto Wess-Zumino models that belong to similar equivalence classes.

To see one of the W_∞ structures, we make use of the field (14), in terms of which Lipatov's kernel is expressed. One easily sees that the following algebra is satisfied for the non-compact case $s = -1$, which has a highest-weight representation and $|2s + 1| = 1$ states:

$$[M_{i_1 j_1}, M_{i_2 j_2}] = \delta_{j_1 i_2} M_{i_1 j_2} - \delta_{j_2 i_1} M_{i_2 j_1}, \quad (17)$$

which is a field-theory realization of a W_∞ algebra. Notice the formal similarity of the algebra (17) to the corresponding one generated by fermion bilinears in two-dimensional large- N_c QCD with adjoint fermions, considered in [14]. The difference is that in our case Lipatov's Hamiltonian pertains to the pure 'glue' sector of quark-quark high-energy scattering processes, and the associated fermion bilinears arise from the mapping of the model to a spin system with doping.

The supersymmetry of the doped theory suggests the existence of another W_∞ structure. This must be associated with the bosonic degrees of freedom of the Ansatz (8). One can construct infinite bosonic symmetries out of these variables, which resemble W_∞ structures. Indeed, the Hamiltonian (6) depends on the composite bosonic bilocal operator $z_i z_j^\dagger$, which transforms like a gauge link variable. In the continuum limit, one can define the bilocal field (in space):

$$\Phi(x, y; t) = z(x, t) z^\dagger(y, t), \quad (18)$$

which satisfies the W_∞ algebra:

$$[\Phi(x, y; t), \Phi(x', y'; t)] = \delta(x - y') \Phi(x, y') - \delta(y' - x) \Phi(y, x'). \quad (19)$$

It should be remarked that the above algebra is *classical* in the sense that it was derived by operators in the model which are constructed so as to obey the canonical commutation relations. Quantum corrections that arise after path integration [10] should in general modify the algebra by appropriate central extensions, as well as non-linear terms [11]. Investigations of these issues falls beyond the scope of the present article.

Notice that this algebra pertains to the field space of the two-dimensional high-energy limit of QCD. In this respect it is a 'target-space' symmetry algebra of the corresponding σ model. It is clear from [2] that there is a double set of mutually-commuting infinite-dimensional Cartan subalgebras in Lipatov's model [1]. However, in contrast to the above discussion, these symmetries associated with the integrability of the model are 'world-sheet' symmetries. The world sheet in this case consists of the physical transverse impact-parameter space of the Reggeized gluons, and its finite size is related to the number of them in a physical high-energy quark-quark scattering process. We expect that the connection between the target-space

and world-sheet pictures in this case is provided in an analogous way to the elevation of world-sheet W_∞ symmetries to target space in the two-dimensional black hole case. In the black-hole case, this association was achieved by appropriate $(1, 1)$ deformations, but it remains to be seen what is the precise form of such operators in our present case. Until this is done, this form of association can only be considered as a conjecture.

Before closing this section, it is useful to investigate the form of the world-sheet symmetry algebras of the present model. Their Cartan subalgebras have been constructed by Fadeev and Korchemsky using Lax operator techniques [2]. Let us first concentrate in the Q_k set, which in their notation consists of operators of the form:

$$Q_k = \sum_{n \geq i_1 \geq \dots \geq i_k} i^k z_{i_1 i_2} z_{i_2 i_3} \dots z_{i_{k-1} i_k} \partial_{i_1} \dots \partial_{i_k}, \quad (20)$$

where $z_{i_1 i_2} \equiv z_{i_1} - z_{i_2}$, $\partial_{i_1} \equiv \partial / \partial z_{i_1}$. A classical w_∞ algebra acting on a holomorphic function is generated by operators of the form:

$$w_n^m \equiv z^m \partial_n, \quad (21)$$

satisfying

$$[w_n^m, w_{n'}^{m'}] = (nn' - m'm) w_{n+n'}^{m+m'} + \dots, \quad (22)$$

where the \dots indicate possible quantum central extensions. Viewing the commutator as a Poisson bracket on a two-dimensional phase space, these are transformations that preserve the phase-space area. The Cartan subalgebra corresponds to the subset with $m = n$, i.e. to an equal number of coordinates and momenta, exactly as happens in (20). One can, therefore, proceed formally to construct the remaining generators of the w_∞ algebra by defining:

$$Q_k^l = \sum_{n \geq i_1 \geq \dots \geq i_1} i^k z_{i_1 i_2} z_{i_2 i_3} \dots z_{i_{k-1} i_k} \partial_{i_1} \dots \partial_{i_k}. \quad (23)$$

These operators can be constructed explicitly in the $s = 0$ case, and then extended to $s = -1$ by a similarity transformation. The required transformation in our case is given by:

$$O_{s=-1} \equiv (z_{12} z_{23} \dots z_{n1})^{-1} O_{s=0} (z_{12} z_{23} \dots z_{n1}). \quad (24)$$

These transformations apply in the case where there are periodic boundary conditions in the finite-size chain. For an infinitely-long doped chain the similarity transformation is provided by:

$$S_{doped}^{-1} = \prod_{i < k} (z_i - z_k). \quad (25)$$

Such similarity transformations are non-unitary and they also appear in Quantum Hall systems, connecting operators pertaining to the Integer and Fractional Quantum Hall Effects [7]. The W algebra pertaining to the operators (23) corresponds to the algebra of a single Landau level in the Quantum Hall case. We expect that the second set, I , should correspond to the W algebra that mixes the Landau levels in the Quantum Hall case, but this remains to be demonstrated.

3 Generalized Heisenberg ferromagnets, compact and non-compact non-linear σ models

The precise nature of these world-sheet W algebras can be investigated if one finds a σ -model representation of the above picture, and uses conformal field theory techniques to construct the generators of the transformations. As we shall argue below, the singular limit $s = 0$ can be considered from the point of view of a non-compact spin problem, which suggests the representation of the theory in terms of a $SL(2, C)$ algebra. Restricting ourselves to the subgroup $SL(2, R)$ and taking into account the extra $U(1)$ symmetry, one can conjecture the form of the σ model that is appropriate for such a system: it is a gauged Wess-Zumino model, based on the group $SL(2, R)/U(1)$. The topological nature of the problem may be captured by an appropriate twisted world-sheet supersymmetry which can be taken to be $N = 2$. To substantiate these claims, but not to prove them rigorously, we now review briefly the situation in the compact spin case, and then continue the results analytically to the non-compact case.

It is well known [6] that Heisenberg spin models may be mapped onto $O(3)$ non-linear σ -models in the limit of large spin s :

$$S_1^2 + S_2^2 + S_3^2 = s(s+1) > 0. \quad (26)$$

As reviewed in [6], the spin Hamiltonian for large spin s corresponds to the Lagrangian

$$L = \frac{1}{2g} \partial_\mu \phi \partial^\mu \phi + \frac{\theta}{8\pi} \epsilon^{\mu\nu} \phi (\partial_\mu \times \partial_\nu \phi) \quad (27)$$

with the conventional normalization constraint $\phi^2 \equiv \phi_1^2 + \phi_2^2 + \phi_3^2 = 1$, and the following identifications of the coupling constant g and the topological angle θ that appears in the antiferromagnetic case:

$$g = \frac{2}{s}, \quad \theta = 2\pi s. \quad (28)$$

We do not discuss further the interesting physics associated with the θ parameter [6]. In addition to the formal derivation of the $O(3)$ non-linear σ model in the limit of large s , there is also evidence that this model describes correctly features of Heisenberg models with small s , as reflected in Fig. 1. For example, both analytical and numerical studies support the maintenance of Néel order in the antiferromagnetic ground state for $s = 1/2$ [6], as assumed in deriving the σ model in the continuum limit.

The two-dimensional $O(3)$ non-linear σ model is well understood: indeed, its exact S -matrix is known. For our purposes, the most useful formulation is as a CP^1 σ model, in which the unit vector ϕ is represented using a two-component complex spinor z_α :

$$\phi = z^* \sigma z \quad ; \quad |z|^2 = 1, \quad (29)$$

where the σ are Pauli matrices, in terms of which the Lagrangian (27) becomes

$$L = \frac{1}{g} [|\partial_\mu z|^2 + (z^* \partial_\mu z)^2] + \frac{\theta}{2\pi} \partial_\mu \epsilon^{\mu\nu} (z^* \partial_\nu z). \quad (30)$$

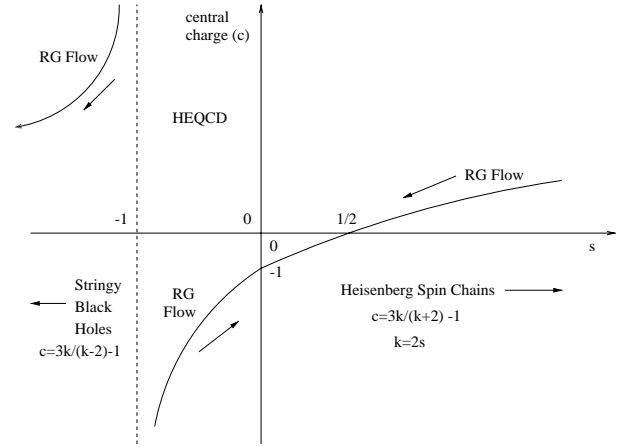


Fig. 1. Map of the space of the gauged compact and non-compact non-linear σ models discussed in this paper, including conventional Heisenberg spin chains with $s \geq 1/2$ that are described by $SO(3)/U(1)$ CP^1 models, and analytic continuations to $s < 0$ that are described by $SO(2, 1)/O(2)$ or $SO(2, 1)/O(1, 1)$ models. High-energy scattering in QCD is described by a twisted supersymmetric version of the limiting cases $s = 0, -1$, and stringy black holes by models with $s < -1$. The vertical coordinate measures the central charge c , and we exhibit the renormalization-group flows towards $s = 1/2$ in the infrared limit of unitary spin models, towards $s = 0$ for the anti-unitary models whose twisted supersymmetric version describes high-energy scattering in perturbative QCD, and away from $s = -1$ for the unitary stringy black-hole models

Insight into this model is obtained by rewriting it in terms of the $U(1)$ gauge field:

$$A_\mu = iz^* \partial_\mu z, \quad (31)$$

using this the Lagrangian (30) can be expressed as:

$$L = \frac{1}{g} |(\partial_\mu + iA_\mu)z|^2 - \frac{i\theta}{2\pi} \partial_\mu \epsilon^{\mu\nu} A_\nu. \quad (32)$$

Solving this model via a saddle-point method, one finds that the z bosons acquire masses:

$$m = \Lambda e^{-2\pi/g} = \Lambda e^{-\pi s}, \quad (33)$$

where Λ is an ultra-violet cutoff. The z bosons are free, apart from interactions via a massless Abelian gauge field with an effective gauge coupling strength:

$$e^2 = 6\pi^2 m^2. \quad (34)$$

This gauge interaction confines the z bosons in the $1 + 1$ -dimensional case, leading to a massive triplet of bound states, in agreement with the exact S -matrix results for the $O(3)$ non-linear σ model [6].

To identify the conformal field theory to which this model corresponds one has to apply finite-size scaling methods in the way studied in [6]. The result of such an analysis indicates that the CP^1 model belongs to the same equivalence class as the $SU(2)$ Wess-Zumino conformal

field theory. As is well known [6], the central charge of the $SU(2)$ Wess-Zumino model is:

$$c = \frac{3k}{(k+2)}, \quad (35)$$

where the level parameter $k = 2s$. Note, in addition, that c is reduced by 1 if a $U(1)$ subgroup is gauged. One frequently considers an ultra-violet limit $\Lambda \rightarrow \infty$ in which the non-linear σ -model coupling $g = 2/s$ also $\rightarrow \infty$ (and hence $k \rightarrow 0$) in such a way that the z -boson mass m and the CP^1 $U(1)$ gauge coupling e remain fixed. However, one can also consider the case in which $s = k/2$ is fixed to be zero, corresponding formally to $g = \infty$, and take the limit $\Lambda \rightarrow \infty$. In this case, m and e become infinite and the only remaining physical field is the $U(1)$ gauge field, which is however completely topological in nature, since it is non-propagating. This interpretation of the $s = k/2 = 0$ theory is supported by the fact that the formula (35) yields a central charge $c = 0$ in this case, corresponding to a topological gauge theory. We therefore identify the $s = 0$ Heisenberg model formally as a topological $U(1)$ gauge theory in the continuum limit.

We now consider the continuation of the above results to $s < 0$. It is clear that the quantum Heisenberg model cannot be represented in terms of Hermitean operators when $-1 < s < 0$, since the quadratic Casimir

$$S_1^2 + S_2^2 + S_3^2 = s(s+1) < 0 \quad (36)$$

for this range of s . One must continue one or more of the spin components $S_{1,2,3}$ to complex values, and the simplest two inequivalent possibilities are to take one or two of the components to be anti-Hermitean. Taking a naive continuum limit in which each of the spin components is replaced by a conventionally-normalized local field variable, the two inequivalent possibilities are:

$$-\phi_1^2 - \phi_2^2 + \phi_3^2 = 1, \quad (37)$$

and

$$\phi_1^2 - \phi_2^2 + \phi_3^2 = 1, \quad (38)$$

where each of the field components $\phi_{1,2,3}$ is understood to be real. These both represent $SO(2,1)$ group manifolds, but with different gaugings. Implementing the manifold constraint by taking ϕ_3 as the dependent variable, one finds:

$$\phi_3 = \sqrt{(1 + \phi_1^2 + \phi_2^2)}, \quad (39)$$

and

$$\phi_3 = \sqrt{(1 - \phi_1^2 + \phi_2^2)} \quad (40)$$

in the two cases. If these models are gauged the corresponding gaugings are with respect to compact $O(2)$ and non-compact $O(1,1)$, respectively. We have no formal proofs, but expect that non-linear σ models on the non-compact manifolds $SO(2,1)/O(2)$ and $SO(2,1)/O(1,1)$ are the continuum field theories corresponding to possible continuations of the spin systems (36) to the range $-1 < s < 0$. The central charges for these models are known to be:

$$c = \frac{3k}{(k-2)} - 1, \quad (41)$$

where the level parameter $k = -2s$ for $s < 0$. The subtraction of unity in (41) reflects the gauging of the non-linear σ model.

The variant corresponding to the high-energy scattering problem should be the non-linear $SO(2,1)/O(2)$ or $SU(1,1)/U(1)$ model. This has the correct local symmetry corresponding to the conservation of the number of Reggeons exchanged in the t channel, i.e., the number of spin variables per lattice site in the Heisenberg spin chain. Moreover, it is known to possess a W_∞ symmetry for generic values of the level parameter $k = -2s$ [11]. As pointed out by Faddeev and Korchemsky [2], Lipatov's model [1] of high-energy scattering can be regarded as a combination of a holomorphic $s = -1$, i.e., $k = 2$, spin chain and an anti-holomorphic $s = 0$, i.e., $k = 0$, spin chain, and these are related by a similarity transformation. We have already argued that the $s = 0$ model is a topological $U(1)$ gauge theory, and shall now argue the same for the $s = -1$ model, based on a previous analysis [11] of the $SU(1,1)/U(1)$ model for $k > 2$, i.e., $s < -1$.

The $k > 2$ $SU(1,1)/U(1)$ model is known [8] to describe a black hole in string theory, with mass proportional to $1/\sqrt{(k-2)}$, as shown in Fig. 1. As can be seen from equation (41), the central charge $c > 2$ in this range of k , and the string black hole becomes critical when $k = 9/4$, since then $c = 26$ in the absence of other degrees of freedom. It has been pointed out [15] that this model is described in the neighbourhood of the singularity at the centre of the black hole by a topological $U(1)$ gauge theory coupled to matter fields (a, b) that parametrize the appearance of space-time coordinates away from the singularity w :

$$\int d^2z \frac{k}{4\pi} (D_z a \cdot D_{\bar{z}} b + w \epsilon^{ij} F_{ij}). \quad (42)$$

In the limit $k \rightarrow 2$, all of space-time is absorbed by the singularity, $a, b \rightarrow 0$, and the theory becomes a pure topological $U(1)$ gauge theory without matter fields. It has further been pointed out that this theory has $N = 2$ supersymmetry, and that the bosonic part of this symmetry algebra includes a $W_\infty \otimes W_\infty$ algebra [15].

Such a topological theory has been constructed in [15] by twisting the $N = 2$ supersymmetric Wess-Zumino model on $SL(2, R)/U(1)$, adding to its stress tensor a derivative of the $U(1)$ current so as to ensure $c = 0$ [16]. It is known that $N = 2$ superconformal theories may be constructed by adding and subtracting a free boson [17]. Unitarity is in general required in those constructions, although it may be relaxed in some cases. The central charge of this supersymmetric coset construction is given by

$$c = \frac{3k}{k-2} \quad (43)$$

yielding the result $c = 0$ as $k = 2s \rightarrow 0^-$, which coincides with the ungauged case. Thus, we now see that the case $s \rightarrow 0$ is free from ambiguities, in the sense that in both the limits $s \rightarrow 0^\pm$ the central charge $c \rightarrow 0$ in an unambiguous way.

However, the limit $s + 1 \rightarrow 0$ is ambiguous, since the central charge of the $s \rightarrow -1^-$ black-hole theory has the

limiting behaviour $c \rightarrow +\infty$, while the central charge in the case $s \rightarrow -1^-$ has the limiting behaviour $c \rightarrow -\infty$. One would like to find some ‘principal value’ prescription to resolve this ambiguity. Crudely speaking, this should be an ‘average’ between the limits $s \rightarrow 1^\pm$ that is a conformal field theory with $c = 0$. Our guiding principle in formulating this is the Bethe-Ansatz approach of [2], where it is observed that the $s = 0$ case is formally isomorphic to the appropriate version of the $s = -1$ case. This prescription can be achieved formally by representing the $s = -1$ case (and the equivalent $s = 0^-$ model) as a topological σ -model field theory on the world sheet, that corresponds to the impact-parameter space in the present case of high-energy QCD. The requisite topological σ model may be constructed by the appropriate twisting [16] of an $N = 2$ supersymmetric σ model, causing the effective central charge of the theory to vanish in the ungauged case ². As we shall see, an important aspect of this construction is the emergence of a cigar-like metric, whose singularity describes the limiting case $s(s+1) = 0$. A nice feature of this type of metric is that it is the limiting non-compact target-space-time case that admits instanton solutions, as we discuss in the next section.

To justify this scenario formally, we pursue in more detail our spin-charge-separation formalism for the description of the antiferromagnet, according to which the magnon sector z is described by the CP^1 continuum field theory. In the absence of fermions, the latter would be equivalent to the $O(3)$ σ model written in terms of the η^α , $\alpha = 1, 2, 3$ variables: $\eta = \bar{z}\sigma^\alpha z$, where the σ^α are the 2×2 Pauli matrices, with $\bar{z}z = \text{const.}$ corresponding to the Casimir condition $\sum_{\alpha=1}^3 \eta^{2,\alpha} = s(s+1)$. As was discussed in [10], there is a corresponding approximate formalism, which is valid in the presence of fermions, in a variant of the Heisenberg chain with next-to-nearest-neighbor interactions. In this case, the Casimir constraint on the magnon fields z reads:

$$\bar{z}z + \frac{1}{G}\psi^\dagger\psi = \text{const.}, \quad (44)$$

where $G \sim t'/J' \rightarrow \infty$ in the model of [10], where the primes denote next-to-nearest-neighbor interactions. This enables one to maintain an approximate connection with the $O(3)$ σ model for antiferromagnets in this formalism.

We now show that this is helpful for identifying the conformal field theory that corresponds to the limiting cases $s(s+1) \rightarrow 0^-$. In the general case with complex spin, we start with the σ model continued analytically to negative s :

$$\frac{1}{s(s+1)} \int d^2z \sum_{i=1}^3 (\partial_\mu \eta^i) g_{ij} (\partial_\mu \eta^j), \quad (45)$$

where the spin variables satisfy:

$$\sum_{i=1}^3 \eta^i g_{ij} \eta^j = s(s+1), \quad (46)$$

² Upon this twisting, the supersymmetric partner fermion fields become BRST ghost fields

and the metric g_{ij} that contracts the spin indices is Minkowskian in the non-compact case, and Euclidean in the compact case. The Casimir factor $s(s+1)$ should be retained as one takes the singular limit $s(s+1) \rightarrow 0$, corresponding to the limits $s \rightarrow 0^-$ and $s = -1^+$, where it becomes a singular constraint that should be solved without making a prior normalization with respect to $s(s+1)$. Defining the variables:

$$w = \frac{\eta_1 + i\eta_2}{a - i\eta_3}, \quad \bar{w} = -\frac{\eta_1 - i\eta_2}{a + i\eta_3}, \quad a \equiv \sqrt{s(s+1)}, \quad (47)$$

where a^2 is *negative* in the non-compact case, at finite s we can map the classical lagrangian (45) onto the following σ model:

$$\frac{\lambda^2}{\pi} \int d^2z \frac{1}{(1+w\bar{w})^2} (\partial_\mu w \partial^\mu \bar{w}), \quad (48)$$

where

$$\lambda^2 = \frac{\pi}{s(s+1)}. \quad (49)$$

This resembles formally a conventional $O(3)$ σ model, although it has negative-definite metric in the non-compact case $s(s+1) < 0$. Formally, the central charge would be $c = 3k/(k+2)$ with $k = 2s$.

We observe that the metric $g(w)$ becomes singular in the limit $s(s+1) \rightarrow 0$. and the theory is topological, although the metric is not singular for other values of $s(s+1)$. However, it should also be noted that the expression of the action in terms of the η variables is not regular at the point $\eta_1 = \eta_2 = 0$. To avoid this problem, as we shall discuss below, we define the theory in the $s(s+1) < 0$ regime through analytic continuation, using the variables (47), in terms of which we construct a σ model with Minkowski metric $g(w) = 1/(1+w\bar{w})$. When expressed in terms of the η variables, the action assumes the linear form (45), up to an overall normalization. However, the metric is singular, and the theory at the core of the singularity is topological for any s such that $s(s+1) < 0$. The limit $s(s+1) \rightarrow 0^-$ may be taken smoothly, with the theory remaining Minkowskian. This limit corresponds precisely to the singularity of the black-hole metric, and the corresponding theory is topological. In that limit, the theory can be rotated without problems to a Euclidean theory that possesses instantons, as we discuss below. This is consistent with the above-mentioned equivalence of the two cases (39),(40) in the limit $s(s+1) \rightarrow 0^\pm$.

To gain formal insight into the nature of the relevant conformal field theory in this limit, we notice that when $s(s+1) = 0$, where the $O(3)$ σ model (48) has a singular metric tensor $g(w, \bar{w}) = 1/(0)^2 \rightarrow \infty$. One may regulate the theory in this limiting case by defining variables:

$$w = \frac{\eta_1 + i\eta_2}{-i\eta_3}, \quad \bar{w} = \frac{\eta_1 - i\eta_2}{i\eta_3}, \quad (50)$$

and keeping $s(s+1)$ arbitrarily small, but non-zero in (46). Then one reproduces (45) in the limit $s(s+1) \rightarrow 0$ using a metric $g(w, \bar{w})$ in (48) of the form:

$$g(w, \bar{w}) = \frac{1}{1 + w\bar{w}}, \quad (51)$$

which is of the cigar-like Euclidean black-hole type discussed in [8]. The latter is known to be described by a non-compact $SL(2, R)/U(1)$ conformal field theory and, as mentioned previously, the limiting case $s+1=0$ corresponds to the singularity of this black hole. The latter is known [8, 16, 15] to be described by a topological world-sheet conformal field theory, obtained from the $N=2$ supersymmetric world-sheet σ -model by a suitable twisting which ensures that $c=0$, as mentioned earlier.

All the above coset constructions require $U(1)$ symmetries. As we have shown in the previous section, the holomorphic sector of Lipatov's spin-chain model, which has $s=-1$, has just such a bosonic symmetry in the limit of a large number of Reggeized gluons, supporting its identification with the $k \rightarrow 2$ limit of the black hole $SU(1, 1)/U(1)$ model.

Thus, we reach the remarkable conclusion that spin systems with $s(s+1) \rightarrow 0^-$, corresponding to complex spin, can be reformulated as topological σ models. In point of fact, as we argue below, the topological symmetry is broken by instanton effects that induce a non-perturbative renormalization-group flow.

4 Renormalization-group flow, instantons and high-energy scattering

We explore in this section the extent to which the understanding obtained above of the topological field-theoretical continuum limit of Lipatov's spin-chain system may cast light on the nature of high-energy scattering and provide, in particular, information on the dependence of the Reggeon intercept on the number of Reggeized gluons. Our main tools in this analysis are the renormalization group and Zamolodchikov's C theorem [20]. We recall first that renormalization-group evolution entails a thinning out of the physical degrees of freedom, which corresponds to a decrease in the central charge c for unitary models. As we discuss later, this theorem requires modification for anti-unitary models such as those relevant to high-energy scattering.

We start by discussing the unitary models in the $s \geq 1/2$ region of Fig. 1, which are described by $SU(2)/U(1)$ non-linear σ models, as discussed in Sect. 3. The effective coupling $g(L)$ increases as the infrared cutoff L is increased:

$$\frac{dg}{d\ln L} = \frac{g^2}{2\pi} \quad ; \quad g(L) \simeq g_0 / [1 - \frac{g_0}{2\pi} \ln L]. \quad (52)$$

Bearing in mind the relation $g = 2/s$, we see that this corresponds to a decrease in the effective spin s , i.e., a decrease in the level parameter $k = 2s$ and hence in the

central charge (35), in agreement with Zamolodchikov's C theorem.

A similar analysis applies in the other unitary region, $s < -1$ corresponding to $k > 2$ for the $SU(1, 1)/U(1)$ non-linear σ model. This region has been discussed elsewhere [15] in connection with string black-hole decay, which is due to higher-genus effects that renormalize the effective action. They provide an absorptive part that is a signature of instability, increase k and hence decrease the black-hole mass, which is proportional to $1/\sqrt{k-2}$. This also corresponds to a decrease in the central charge c , as given by (41), in agreement with the C theorem. This higher-genus decay effect can be represented by instantons in the effective lowest-genus theory, since these describe transitions between string black holes of different masses, i.e., different values of k and c [15].

The discussion of theories with $1/2 > s > -1$ is more complicated, because they are anti-unitary, a property traceable to the fact that in high-energy scattering one is calculating the energy dependence

$$s^\epsilon = e^{\epsilon \log s} \quad : \quad \langle H \rangle = \epsilon \quad (53)$$

rather than a normal unitary evolution e^{iHt} . The latter is related to the former (53) by $H \rightarrow iH$, which corresponds to a change in sign in $c = \langle TT \rangle$. Under these circumstances, Zamolodchikov's C theorem does not apply [20]. However, even in anti-unitary theories the renormalization-group flow must be such as to thin out the physical degrees of freedom [21].

Symmetry breaking usually arises because the unbroken phase of the theory has more degrees of freedom than the broken phase, as is, for instance, the case for the topological phase of the $N=2$ σ models corresponding to a Wess-Zumino theory on $SL(2, R)/U(1)$. In such models, the topological phase consists of an infinity of non-propagating topological modes of the string. The latter couple to the propagating string modes as a result of non-perturbative conformal invariance [15]. This theory has instantons (holomorphic maps) whose suppression is not bounded away from zero³. These induce extra logarithmic scale dependences in correlation functions, vacuum energies, etc., which depend on the size of the world sheet. They imply a breaking of the topological symmetry and a thinning of the physical degrees of freedom of the system.

We now argue that a similar instanton effect occurs in the case of high-energy scattering, starting from the conventional representation of the $s > 0$ spin system in the continuum limit as an $O(3)$ σ model. This representation holds exactly only in the limit of large s , but it will be sufficient for our purposes. Denoting by $\eta_i : i = 1, 2, 3$ the mean-spin variable, with $|\eta| = 1$, the action of the $O(3)$ σ model can be written in terms of the complex variables (47). The action $\int d^2x (\partial_\mu \eta_i)^2$ can then be written in the form (48), where the metric $g(w)$ is given by:

$$g(w, \bar{w}) = \frac{1}{(1 + |w|^2)^2}. \quad (54)$$

³ The metric of the $SL(2, R)/U(1)$ Euclidean black-hole target space is actually the limiting case in which the instantons are unsuppressed, as a result of the non-compact moduli space

The σ model (48) with the metric (54) has instanton solutions:

$$w(z) = u + \frac{\rho}{z - z_0} \quad (55)$$

with winding number $n = 1$ ⁴, which describe transitions between the different topological sectors of the theory, that are classified by the θ term of the model. These instanton transitions reduce the central charge c by reducing $k = 2s$ towards zero from above, as illustrated in Fig. 1.

We now extend this discussion to include non-compact target spaces. To this end, we generalize the metric $g(w)$ (54) to:

$$g(w, \bar{w}) = \frac{1}{(1 + |w|^2)^q} \quad (56)$$

with q arbitrary but real. Instanton solutions of the classical action exist only for $q > 1/2$. For $q > 1$ the target space of the σ model is compact, and one has the conventional instanton. We have argued in the previous section (51) that the case $q = 1$ corresponds to the conformal field theory describing the limit $s(s + 1) \rightarrow 0$. The instanton action is finite [19] for $q = 1$:

$$S_I = a \frac{1}{3} \lambda^2, \quad (57)$$

where a is a numerical coefficient that depends on the regularization scheme. This implies that the instanton contribution in the correlation functions of the model will be weighted by g_I , where

$$g_I = e^{-a \frac{\lambda^2}{3}}. \quad (58)$$

In compact σ models, the anti-instantons have zero action, and as such they do not contribute to correlation functions. On the other hand, in the black-hole conformal field theory, the anti-instantons make non-trivial contributions to the correlation functions of the model [19]. Their effects may be summarized by adding an effective vertex:

$$V_{\bar{I}} \sim -g_{\bar{I}} \int d^2\sigma g(w) \bar{\chi} \chi \partial_\mu^2 (g(w, \bar{w}) \bar{\chi} \chi), \quad (59)$$

where χ denotes the spin-0 fermions of the twisted $N = 2$ supersymmetric black-hole σ model.

The observables in the topological $q = 1$ σ model do not depend directly on λ^2 . In view of the above-mentioned regularization-scheme dependence, therefore, one may consider g_I (58) as the true renormalized coupling constant of the model [19]. As already mentioned, the value $q = 1$ [19] marks the border line between the compact and non-compact target-space cases, where the moduli space of the instantons is non-compact. In the topological version of the $SL(2, R)/U(1)$ model, the instanton action is finite and the instantons constitute relevant operators, as far as the breaking of conformal invariance is concerned [18, 19]. Notice that in the limit $s(s + 1) \rightarrow 0^+$, which for negative

s occurs for $s < -1$, the positive instanton action (57) becomes infinite, and the coupling constant $g_I \rightarrow 0$ (58). On the other hand, in the region where $s(s + 1) < 0$, the instanton contributions to the correlation functions are not suppressed, and in fact diverge as $s(s + 1) \rightarrow 0^-$. In this domain of the renormalization-group flow, the instanton transitions therefore occur very rapidly.

The presence of instanton transitions leads, as we show below, to a breaking of the topological symmetry [19] in the sense of a false vacuum⁵. The presence of the false vacuum implies that the phase $s(s + 1) < 0$ is unstable, driving the theory to the limiting case $s = 0$, which is equivalent to our ‘principal value’ version of the case $s = -1$.

To this end, we first review the breaking of the topological symmetry by instantons in this class of theory. First, we note that in our case the existence of instantons is associated with the Berry-phase term in the spin model, as discussed in [9]. For our purpose, we note that this term becomes, in the case of $s = 0$, just the complex-structure term of the topological σ model, i.e. in terms of the w variables (47):

$$S_B = \int d^2z g(w) (\bar{\partial} w \partial \bar{w} - \bar{\partial} \bar{w} \partial w) \quad (60)$$

with the same normalization coefficient as the kinetic term. In terms of the η_i spin variables, this yields a term:

$$\frac{1}{s(s + 1)} \int d^2z \epsilon_{\alpha\beta} \frac{\eta_1}{\eta_3} \partial_\alpha \eta_2 \partial_\beta \eta_3, \quad (61)$$

which, using the Casimir constraint to express η_3 in terms of $\eta_{1,2}$, becomes a total derivative:

$$\frac{1}{s(s + 1)} \int d^2z \epsilon_{\alpha\beta} \frac{1}{\eta_1^2 + \eta_2^2} \partial_\alpha (\eta_2^3) \partial_\beta \eta_1. \quad (62)$$

We note that the Berry-phase term (61) differs from the conventional Berry-phase spin term by the factor $1/\eta_3$. This extra power of the spin variable is essential in this singular limit to guarantee the correct dimensionality in spin space. To understand this, note that, in the non-singular case, normalization of the spin variable by division by the square root of the non-vanishing Casimir coefficient is possible. However, this is not possible in the singular limit we are considering here. In this case, the point $\eta_i = 0$ for all $i = 1, 2, 3$ is allowed, in contrast to the non-singular positive-spin case. The Berry-phase term has to be regular at this point, since it is a finite topological invariant, the winding number, and the only way to achieve this is to normalize by dividing with $1/\eta_3 s(s + 1)$. The guiding principle is to construct a continuous version of the Berry-phase term that renders the instanton deformations of the σ model relevant operators. At this stage, we still lack a first-principles construction of the complex-structure term from the underlying statistical model, but

⁴ It is straightforward to incorporate solutions with higher winding number, that have a proportionality constant n in front of the instanton action

⁵ The presence of a non-zero Witten index in the twisted $N = 2$ supersymmetric σ model implies that supersymmetry can only be broken in the sense of a false vacuum

the above heuristic arguments for its form are sufficient for our purposes.

The existence of such a topological term guarantees that the instantons have finite action, in contrast to the anti-instantons whose action diverges logarithmically with the area of the world-sheet. Thus, instanton-anti-instanton configurations can lead to extra logarithmic dependences in correlation functions, that can affect the conformal invariance. The doping Ansatz we adopted earlier can supersymmetrize the σ model, as appropriate for its topological nature in the limit $s = 0$. For an analysis of instantons in this supersymmetric version see [19]. The important point is that the instantons result [22] in a renormalization of the Wess-Zumino level parameter of the σ -model $k(=2s) \rightarrow k(\ln(\Lambda/\ell))$, where Λ (ℓ) is an infrared (ultraviolet) world-sheet renormalization-group length scale.

To understand this, we first note that the instanton-anti-instanton vertices introduce new terms into the effective action. Making a derivative expansion of the instanton vertex and taking the large- k limit, i.e. restricting our attention to instanton sizes $\rho \simeq \ell$, these new terms acquire the same form as the kinetic terms in the σ model, thereby corresponding to a renormalization of the effective level parameter in the large $k(=2s)$ limit [22], related to the $SL(2, R)/U(1)$ coset black-hole model [8]:

$$k(=2s) \rightarrow k - 2\pi k^2 d' \quad ; \quad d' \equiv g^I g^{\bar{I}} \int \frac{d|\rho|}{|\rho|^3} \frac{\ell^2}{[(\rho/\ell)^2 + 1]^{\frac{k}{2}}}, \quad (63)$$

where ρ denotes the collective coordinate of the instantons (55). If other perturbations are ignored, the instantons are irrelevant deformations and conformal invariance is maintained. However, in the $SL(2, R)/U(1)$ coset black-hole model, there exist matter deformations, $T_0 \int d^2 z \mathcal{F}_{-\frac{1}{2}, 0, 0}^{c, c}$, with T_0 assumed positive in the $SL(2, R)$ notation of [23], which change drastically the situation [22]. Similar matter excitations also appear in the spectrum of the exact solutions of the Baxter equation for the spin model of high-energy QCD of [2], so we need to take them into account.

The matter deformations induce extra logarithmic infinities in the shift (63), which are visible in the dilute-gas and weak-matter approximations. In this case, there is a contribution to the σ -model effective action of the form:

$$S_{eff} \ni -T_0 \int d^2 z d^2 z' \langle \mathcal{F}_{-\frac{1}{2}, 0, 0}^{c, c}(z, \bar{z}) V_{I\bar{I}}(z', \bar{z}') \rangle, \quad (64)$$

where $V_{I\bar{I}}$ denotes the instanton-anti-instanton deformation. Using the explicit form of the matter vertex \mathcal{F}

$$\begin{aligned} \mathcal{F}_{-\frac{1}{2}, 0, 0}^{c, c} &= \frac{1}{\sqrt{1 + |w|^2}} \frac{1}{\Gamma(\frac{1}{2})^2} \\ &\times \sum_{n=0}^{\infty} \left\{ 2\psi(n+1) - 2\psi\left(n + \frac{1}{2}\right) + \right. \\ &\left. + \ln(1 + |w|^2) \right\} (\sqrt{1 + |w|^2})^{-n} \end{aligned} \quad (65)$$

given by $SL(2, R)$ symmetry [23], it is straightforward to isolate a logarithmically-infinite contribution to the ki-

netic term in the σ model, associated with infrared infinities on the world sheet. These are expressible in terms of the world-sheet volume $V^{(2)}/\ell^2 = \Lambda^2/\ell^2$, the latter measured in units of the ultraviolet cut-off ℓ :

$$\begin{aligned} S_{eff} &\ni -T_0 g^I g^{\bar{I}} \int d^2 z' \int \frac{d\rho}{\rho} \left(\frac{\ell^2}{\ell^2 + \rho^2} \right)^{\frac{k}{2}} \\ &\times \int d^2 z \frac{1}{|z - z'|^2} \frac{1}{1 + |w|^2} \partial_{z'} w(z') \partial_{\bar{z}'} \bar{w}(z') + \dots, \\ &\propto -T_0 g^I g^{\bar{I}} \ln \frac{\Lambda^2}{\ell^2} \int d^2 z' \frac{1}{1 + |w|^2} \partial_{z'} w(z') \partial_{\bar{z}'} \bar{w}(z'). \end{aligned} \quad (66)$$

The logarithmic scale dependence can be absorbed in a shift of k : $k_{ren} = k - T_0 g^I g^{\bar{I}} \ln(\Lambda/\ell)$. The net result of such a renormalization is to reduce the magnitude of k [22]. The central charge $c = 3k/(k-2)$ of the twisted model changes as follows:

$$\frac{\partial}{\partial t} c = -\frac{6}{(k-2)^2} \frac{\partial}{\partial t} k \quad ; \quad t \equiv \ln(\Lambda/\ell) \quad (67)$$

Thus, the rate of change of c is *opposite* to that of k . Therefore, by reducing k , one increases the central charge. Since, for $s \rightarrow 0^-$ ($k \rightarrow 0^+$), $c \rightarrow 0$, one then observes from (67) that, under the instanton-induced renormalization-group flow, the central charge is driven towards $c = 0^-$, in the limit $t \rightarrow \infty$, as $c \simeq -6/(T_0 g^I g^{\bar{I}} t)$.

We note now that the resulting vacuum energy can be found by computing the vertex operator of an anti-instanton in the dilute-gas approximation for a σ -model deformation corresponding to an instanton vertex operator. The result to leading order in the instanton-anti-instanton coupling is:

$$\begin{aligned} E_{vac}^{I\bar{I}} &= \langle V_{\bar{I}} \rangle_I \\ &= -g^{\bar{I}} \int d^2 x_1 \partial_{x_2}^2 \langle O(x_1) O(x_2) \rangle_{|x_1 \rightarrow x_2}. \end{aligned} \quad (68)$$

Recalling that the dominant anti-instanton configurations have sizes $\rho \simeq \ell$, we can use (68) to estimate that in the infrared limit when $\Lambda/\ell \gg u$

$$E_{vac} = -16\pi^2 g^I g^{\bar{I}} \frac{V^{(2)}}{\ell^2} [\ln(\Lambda/\ell) + O(1)], \quad (69)$$

where $V^{(2)}$ is the world-sheet volume.

At this stage, we appear to have some logarithmic dependence as a result of instanton configurations. However, as was argued in [19], upon summing over an arbitrary number of instanton-anti-instanton pairs in the model, the logarithmic infrared divergences in (69) disappear, and the system is equivalent to a Coulomb-gas/sine-Gordon model, in a similar spirit to the compact $O(3)$ case, the formal difference from the latter being that the rôle of instantons in that case is now played by the instanton-anti-instanton pairs.

The details of the analysis can be found in [19], and we describe here only the basic results that are relevant for our purposes. When one maps the system, resummed over

an arbitrary number of instantons and anti-instantons, to a Coulomb gas, the resulting vacuum energy (69) may be re-calculated using the free massive-fermion representation [19], with action:

$$S_{eff} = \frac{1}{\pi} \int d^2x \{ \bar{\psi} \gamma_\mu \partial_\mu \psi + m \bar{\psi} \psi \}, \quad (70)$$

where the mass $|m| = \frac{2\pi}{\ell} \sqrt{8\pi} \sqrt{g_I g_{\bar{I}}}$, whose inverse plays the rôle of a world-sheet infrared cut-off for the system. The resulting vacuum energy is now given by:

$$E_{vac} = \frac{m^2}{2\pi} V^{(2)} \log \frac{m\ell}{2} \\ \propto V^{(2)} g_I g_{\bar{I}} (\log(g_I g_{\bar{I}}) + \text{const}), \quad (71)$$

which shows that the model has finite vacuum energy when resummed over the instanton-anti-instanton pairs, which still break the topological symmetry [19]. We note that the vacuum energy becomes zero in the limit where $s(s+1) \rightarrow 0$ such that $s(s+1) > 0$, and the topological symmetry at the singularity is restored.

In the limit of a large number of Reggeized gluons, the spatial size of the system, corresponding to the volume of the impact-parameter space, can be related to the number of Reggeons⁶: $V^{(2)} \propto N_g$. Thus the vacuum energy of our problem exhibits a non-trivial dependence on the number of Reggeized gluons N_g in the scattering process, given in the large- N_g limit by:

$$E_{vac} \propto N_g, \quad (72)$$

where we have used fact that the world-sheet volume is proportional to N_g . Taking into account also the relation [1] between E_{vac} and the Regge intercept j : $E_{vac} = 1 - j = -\Delta$, we see that the Regge intercept varies linearly with increasing N_g , at least for large N_g . A similar result has also been argued on the basis of a Hartree-Fock approximation to Lipatov's Hamiltonian [4]. The sign of the vacuum energy and hence the shift in the Regge intercept is currently ambiguous in our approach, because the instanton coupling constant (58) has a regularization-scheme dependence [19], and hence should be considered as arbitrary within our approach. For positive coefficients a and $s(s+1) \rightarrow 0^-$, the instanton coupling $g_I \rightarrow \infty$, and one would obtain positive (infinite) vacuum energy (71). However, there exist regularization schemes such that a is proportional to $s(s+1)$ in such a way so that g_I is finite, and even smaller than one. In the framework of high-energy QCD, such ambiguities may be associated with the renormalization-group running of the coupling constant of the system of N_g gluons, which according to [4] could be responsible for the appearance of negative ground state energies in the Hartree-Fock approximation. In contrast, in the fixed coupling constant approach to multicolour QCD, on which the integrable model analysis of [2] is based, the energy comes out positive, implying the instability of

multicolour states, which thus become irrelevant at high energies.

We conclude that the instantons induce an instability and break the topological symmetry [19] via a false vacuum. This results in a tendency of the $-1 < s < 0$ system to flow towards the $s = 0$ ground case. The appropriate choice of vacuum state for the ambiguous case $s = -1$ is then fixed by the requirement of isomorphism to the holomorphic sector, as argued in [2].

5 Conclusions and prospects

We have analyzed in this paper the symmetries of Lipatov's model for high-energy scattering in QCD, and used them to motivate a proposal for the conformal field theory that should describe the continuum limit of Lipatov's model corresponding to the exchange of a large number of Reggeized gluons in the t channel. Arguing by analogy with the known correspondence between compact Heisenberg spin-chain models and non-linear $O(3)$ σ models, we have suggested that Lipatov's model may correspond to a limiting case of a non-compact $SL(2, R)/U(1)$ σ model. An analysis of instantons helps to explain the appearance of this limiting model, which is a topological field theory analogous to that describing the core of a 1+1-dimensional string black hole. It possesses an $N = 2$ supersymmetric algebra that includes the $W_\infty \otimes W_\infty$ bosonic algebra previously identified in Lipatov's model. Formal support of our proposal for identifying the conformal field theory underlying Lipatov's model as the $SL(2, R)/U(1)$ model is provided by the observation reported in the second paper of [2], that the exact solution of the Baxter equation for the $N_g = 2$ Reggeon state bears a remarkable similarity to the spectrum of the $SL(2, R)/U(1)$ coset conformal field theory. Our spin-charge-separation Ansatz may extend this similarity to the multi-Reggeon case as well.

Many aspects of our analysis are heuristic, and merit further study. These include the validity of the 'hole-regulator' scheme that we have proposed, the quantum corrections to the $W_\infty \otimes W_\infty$ symmetry algebra that we have identified, details of its elevation from 'world sheet' to 'target space', and the representation of the second W_∞ algebra. The relation of the non-compact spin-chain and σ models should be clarified, as has previously been done for the compact spin-chain models and $O(3)$ σ models. Also, the rôle of instantons in non-compact σ models merits further investigation.

We hope that our proposal may open the way to a more powerful tool-box for analyzing high-energy scattering in QCD. Field-theoretical techniques may allow the consequences of both t - and s -channel unitarity to be investigated more thoroughly, via the string topological diagram expansion and the power of conformal field theory.

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⁶ We are back to the half-filled case, where the charge (hole) excitations acquire a topological nature. In the $s = 0$ case, this is also true for the spin excitations as well

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